MATH2060B Tutorial 12

9.3 Q14 Show that if the partial sums s_n of the series $\sum_{k=1}^{\infty} a_k$ satisfy $|s_n| \le Mn^r$ for some $r<1$, then the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Recall, Abel's lemma states: if (x_n), (y_n) are sequences in $\mathbb{R}$, let (s_n) be the sequence of partial sums of $\sum y_n$. Then if $m>n$,

$$\sum_{k=n+1}^{m} x_k y_k = x_m s_m - x_{n+1} s_n + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k .$$

Pf Apply Abel's lemma to $x_n = \frac{1}{n}$ & $y_n = a_n$. For $m>n$,

$$\sum_{k=n+1}^{m} \frac{a_k}{k} = \frac{s_m}{m} - \frac{s_n}{n+1} + \sum_{k=n+1}^{m-1} \frac{1}{k(k+1)} s_k .$$

The given condition on (s_n) implies

$$\left| \sum_{k=n+1}^{m} \frac{a_k}{k} \right| \le \frac{Mm^r}{m} + \frac{Mn^r}{n+1} + \sum_{k=n+1}^{m-1} \frac{Mk^r}{k(k+1)}$$

$$\le Mm^{r-1} + Mn^{r-1} + M \sum_{k=n+1}^{m-1} \frac{1}{k^{2-r}} .$$

Note that

(1) Because $r-1<0$, $Mm^{r-1}, Mn^{r-1} \to 0$ as $m,n \to \infty$.

(2) The series $\sum \frac{1}{k^{2-r}}$ is a p-series for $p = 2-r > 1$, which is convergent. So $\sum_{k=n+1}^{m-1} \frac{1}{k^{2-r}} \to 0$ as $m,n \to \infty$.

These imply that $\left| \sum_{k=n+1}^{m} \frac{a_k}{k} \right| \to 0$ as $m,n \to \infty$, and hence by Cauchy Criterion, the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent. $\square$

9.4 Q1(e) Discuss the convergence and uniform convergence of the series $\sum f_n$, where $f_n(x) = \frac{x^n}{x^n+1}$, $x \geq 0$.

Sol

① Observe that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & , \quad 0 \leq x < 1 \\ \frac{1}{2} & , \quad x = 1 \\ 1 & , \quad x > 1 \end{cases}.$$

So $\sum f_n$ diverges on $[1, \infty)$.

② Now, for $a<1$,

$$f_n(x) \leq x^n \leq a^n \qquad \forall x \in [0,a].$$

Since the geometric series $\sum a^n$ is convergent for $a<1$, Weierstrass M-test implies that $\sum f_n$ converges uniformly on $[0,a]$.

Because any $x \in [0,1)$ is contained in some $[0,a] \subset [0,1)$, this also shows that $\sum f_n$ is convergent on $[0,1)$.

③ $\sum f_n$ is not uniformly convergent on $[0,1)$: if it were, then Cauchy criterion (with $\varepsilon = \frac{1}{3}$) implies there exists $N \in \mathbb{N}$ such that $m > n \geq N \Rightarrow \left|\sum_{k=n}^{m} f_k(x)\right| < \frac{1}{3}$, $\forall x \in [0,1)$.

In particular, $f_n(x) < \frac{1}{3}$ for all $n \geq N$, $x \in [0,1)$ because all f_k are non-negative.

However, for any $n \in \mathbb{N}$, let $x_n = \frac{1}{2^{1/n}} \in [0,1)$,

$$f_n(x_n) = \frac{1/2}{1/2+1} = \frac{1}{3}.$$ This is a contradiction and hence $\sum f_n$ is not uniformly convergent on $[0,1)$.

9.4 Q5 Show that the radius of convergence R of the power series $\sum a_n x^n$ is given by $\lim \left|\frac{a_n}{a_{n+1}}\right|$ whenever this limit exists (in $[0,\infty]$).

Pf Let $L = \lim \left|\frac{a_n}{a_{n+1}}\right|$.

Case 1 $0 < L < \infty$. If $|x| < L$, then

$$\lim_{n\to\infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < L \cdot \frac{1}{L} = 1.$$

On the other hand if $|x| > L$, same argument shows that

$$\lim_{n\to\infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} > 1.$$

Ratio test implies that $\sum a_n x^n$ is (i) convergent when $|x| < L$, and (ii) divergent when $|x| > L$.

Using Cauchy-Hadamard theorem, (i) $\Rightarrow$ $L \leq R$ and (ii) $\Rightarrow$ $R \leq L$. So $L = \lim \left|\frac{a_n}{a_{n+1}}\right|$ is the radius of convergence.

Case 2 $L = 0$. Want to show that $\sum a_n x^n$ diverges for $|x| > \delta$, for any given $\delta > 0$. Using the definition of $L=0$, there exists $K \in \mathbb{N}$ such that $\forall n \geq K$,

$$|a_n| < |a_{n+1}| \delta < |a_{n+1} x|.$$

This implies that $|a_n x^n| < |a_{n+1} x^{n+1}|$, so the absolute values of the terms of the series $\sum a_n x^n$ is strictly increasing for $n \geq K$, and does not converge to zero as $n \to \infty$. So the power series diverges for $|x| > \delta$.

Case 3 $L = \infty$. By definition of L, for any $M > 0$, $\exists K \in \mathbb{N}$ such that $|a_n| \geq M |a_{n+1}|$ for all $n \geq K$.

So for any $|x| \leq \frac{M}{2}$, and $n \geq K$

$$|a_n x^n| \leq \frac{|a_{n-1}|}{M} \cdot |x^n| \leq \dots \leq \frac{|a_K|}{M^{n-K}} \cdot |x^n| \leq \frac{M^K |a_K|}{2^n}.$$

$\Rightarrow$ $\sum_{n=K}^{\infty} a_n x^n$ converges by comparing with the geometric series $M^K |a_K| \sum_{n=K}^{\infty} \frac{1}{2^n}$.

Thus $\sum_{n=0}^{\infty} a_n x^n$ also converges for $|x| \leq \frac{M}{2}$. But since $M > 0$ is arbitrary, the power series converges at any $x \in \mathbb{R}$, i.e. $R = \infty = L$. $\square$

Remark Consider the sequence (a_n) with $a_{2k} := \frac{1}{2}$ and $a_{2k+1} := 2$. Then

(a) $\left|\frac{a_n}{a_{n+1}}\right| = 4$ for odd n & $= \frac{1}{4}$ for n even,

so $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ does not exist.

(b) But $\limsup (|a_n|^{1/n}) = \lim_{n\to\infty} |a_n|^{1/n} = 1$. So the radius of convergence of $\sum a_n x^n$ is 1.

(c) In fact $\limsup \left|\frac{a_n}{a_{n+1}}\right| = 4$ and $\liminf \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{4}$, so both are $\neq$ radius of convergence of $\sum a_n x^n$.